QUADRATIC STOCHASTIC OPERATORS: RESULTS AND OPEN PROBLEMS

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Abstract. The history of the quadratic stochastic operators can be traced back to work of S.Bernshtein (1924). During more than 80 years this theory developed and many papers were published. In recent years it has again become of interest in connection with numerous applications to many branches of mathematics, biology and physics. But most results of the theory were published in non English journals, full text of which are not accessible. In this paper we give a brief description of the results and discuss several open problems.

Keywords. Quadratic stochastic operator, fixed point, trajectory, Volterra and non-Volterra operators, simplex.

1 Introduction

Quadratic stochastic operator (QSO) was first introduced in [1]. A QSO has meaning of a population evolution operator, which arises as follows. Consider a population consisting of m species. Let $x^0 = (x_1^0, ..., x_m^0)$ be the probability distribution of species in the initial generations, and $P_{ij,k}$ the probability that individuals in the ith and jth species interbreed to produce an individual k. Then the probability distribution $x' = (x_1', ..., x_m')$ (the state) of the species in the first generation can be found by the total probability i.e.

$$x_k' = \sum_{i,j=1}^{m} P_{ij,k} x_i^0 x_j^0, \quad k = 1, ..., m. \tag{1}$$

This means that the association $x^0 \to x'$ defines a map V called the evolution operator. The population evolves by starting from an arbitrary state x^0, then passing to the state $x' = V(x)$ (in the next "generation"), then to the state $x'' = V(V(x))$, and so on. Thus states of the population described by the following dynamical system

$$x^0, \quad x' = V(x), \quad x'' = V^2(x), \quad x''' = V^3(x), ..., \tag{2}$$

where $V^n(x) = V(V(...V(x))...)$ denotes the n times iteration of V to x.

Note that V (defined by (1)) is a non linear (quadratic) operator, and it is higher dimensional if $m \geq 3$. Higher dimensional dynamical systems are important but there are relatively few dynamical phenomena that are currently understood [2].

The main problem for a given dynamical system (2) is to describe the limit points of $\{x^{(n)}\}_{n=0}^{\infty}$ for arbitrary given $x^{(0)}$. In this paper we discuss recently obtained results on the problem, also give several open problems related to theory of QSOs.

2 Definitions

The quadratic stochastic operator (QSO) is a mapping of the simplex.

$$S^{m-1}=\left\{x=(x_1,...,x_m)\in \mathbf{R}^m : x_i\geq 0, \sum_{i=1}^m x_i=1\right\} \tag{3}$$

into itself , of the form

$$V: x_k'=\sum_{i,j=1}^m P_{ij,k}x_ix_j, \quad k=1,...,m, \tag{4}$$

where $P_{ij,k}$ are coefficients of heredity and

$$P_{ij,k}\geq 0, \quad P_{ij,k}=P_{ji,k}, \quad \sum_{k=1}^m P_{ij,k}=1, (i,j,k=1,....,m). \tag{5}$$

Thus each quadratic stochastic operator V can be uniquely defined by a cubic matrix $\mathbf{P}=(P_{ij,k})_{i,j,k=1}^n$ with conditions (5).

Note that each element $x\in S^{m-1}$ is a probability distribution on $E=\{1,...,m\}$.

For a given $x^{(0)}\in S^{m-1}$ the trajectory (orbit)

$$\{x^{(n)}\}, \quad n=0,1,2,... \ \text{ of } \ x^{(0)}$$

under the action of QSO (4) is defined by

$$x^{(n+1)}=V(x^{(n)}), \quad \text{where} \quad n=0,1,2,...$$

One of the main problem in mathematical biology consists in the study of the asymptotical behavior of the trajectories. The difficulty of the problem depends on given matrix $\mathbf{P}$.

3 The Volterra operators

A Volterra QSO is defined by (4), (5) and the additional assumption

$$P_{ij,k}=0, \quad \text{if} \quad k\notin\{i,j\}, \ \forall i,j,k\in E. \tag{6}$$

The biological treatment of condition (6) is clear: The offspring repeats the genotype of one of its parents.

In paper [10] the general form of Volterra QSO

$$V: x=(x_1,...,x_m)\in S^{m-1} \ \rightarrow \ V(x)=x'=(x_1',...,x_m') \ \in S^{m-1}$$

is given

$$x_k'=x_k\left(1+\sum_{i=1}^m a_{ki}x_i\right), \quad k\in E \tag{7}$$

where

$$a_{ki}=2P_{ik,k}-1 \ \text{ for } i\neq k \text{ and } \ a_{ii}=0, \ i\in E. \tag{8}$$

Moreover

$$a_{ki}=-a_{ik} \ \text{ and } \ |a_{ki}|\leq 1.$$

Denote by $A = (a_{ij})_{i,j=1}^m$ the skew-symmetric matrix with entries (8).

Let $\{x^{(n)}\}_{n=1}^\infty$ be the trajectory of the point $x^0 \in S^{m-1}$ under QSO (7). Denote by $\omega(x^0)$ the set of limit points of the trajectory. Since $\{x^{(n)}\} \subset S^{m-1}$ and S^{m-1} is compact, it follows that $\omega(x^0) \neq \emptyset$. Obviously, if $\omega(x^0)$ consists of a single point, then the trajectory converges, and $\omega(x^0)$ is a fixed point of (7). However, looking ahead, we remark that convergence of the trajectories is not the typical case for the dynamical systems (7). Therefore, it is of particular interest to obtain an upper bound for $\omega(x^0)$, i.e., to determine a sufficiently "small" set containing $\omega(x^0)$.

Denote

$$\mathrm{int}S^{m-1} = \{x \in S^{m-1} : \prod_{i=1}^m x_i > 0\}, \quad \partial S^{m-1} = S^{m-1} \setminus \mathrm{int}S^{m-1}.$$

Definition 1. A continuous function $\varphi : \mathrm{int}S^{m-1} \to R$ is called a Lyapunov function for the dynamical system (7) if the limit $\lim_{n\to\infty} \varphi(x^{(n)})$ exists for any initial point x^0.

Obviously, if $\lim_{n\to\infty} \varphi(x^{(n)}) = c$, then $\omega(x^0) \subset \varphi^{-1}(c)$. Consequently, for an upper estimate of $\omega(x^0)$ we should construct the set of Lyapunov functions that is as large as possible.

In [3],[10]- [14], [45] the theory of QSOs (7) was developed by using theory of the Lyapunov function and tournaments.

The following results are known:

Theorem 1. [10],[14] *For the Volterra QSO (7) the following assertions hold*

1) *For the dynamical system (7) there exists a Lyapunov function of the form* $\varphi_p(x) = x_1^{p_1}...x_m^{p_m}$, *where* $p_i \geq 0$, $\sum_{i=1}^m p_i = 1$ *and* $x = (x_1, ..., x_m) \in \mathrm{int}S^{m-1}$.

2) *If there is* $r \in \{1, ..., m\}$ *such that* $a_{ij} < 0$ *(see (8)) for all* $i \in \{1, ..., r\},\ j \in \{r+1, ..., m\}$ *then* $\varphi(x) = \sum_{i=r+1}^m x_i,\ x \in S^{m-1}$ *is a Lyapunov function for QSO (7).*

3) *There are Lyapunov functions of the form*

$$\varphi(x) = \frac{x_i}{x_j},\ i \neq j,\ x \in \mathrm{int}S^{m-1}.$$

Problem 1. *Does there exist another kind of Lyapunov function for QSO (7)?*

The next theorem related to the set of limit points of QSO (7).

Theorem 2. [10],[14] 1) *If* $x^{(0)} \in \mathrm{int}S^{m-1}$ *is not a fixed point (i.e.* $V(x^{(0)}) \neq x^{(0)}$*), then* $\omega(x^0) \subset \partial S^{m-1}$.

2) *The set* $\omega(x^0)$ *either consists of a single point or is infinite.*

3) *If QSO (7) has an isolated fixed point* $x^* \in \mathrm{int}S^{m-1}$ *then for any initial point* $x^{(0)} \in \mathrm{int}S^{m-1} \setminus \{x^*\}$ *the trajectory* $\{x^{(n)}\}$ *does not converge.*

A skew-symmetric matrix A is called transversal if all even order leading (principal) minors are nonzero.

A Volterra QSO V is called transversal if the corresponding skew-symmetric matrix A is transversal [14],[18],[36].

Problem 2. *Define concept of transversality for arbitrary QSO and find necessary and sufficient condition on matrix* $\mathbf{P} = (P_{ij,k})$ *of a QSO under which the QSO is a transversal.*

Note that if a Volterra QSO is transversal then the set $X = \{x \in S^{m-1} : V(x) = x\}$ of fixed points is a finite set [14].

Let $U \equiv U_X$ be a neighborhood of the set X and $\{x^{(n)}\}$ be an arbitrary trajectory. Denote

$$n_U = \left| \{j = 1, ..., n : x^{(j)} \in U\} \right|,$$

where $|M|$ denotes the number of elements in M.

Then it is known that

$$\lim_{n\to\infty} \frac{n_U}{n} = 1$$

i.e. the trajectory a despondent part of the time will stay in the neighborhood of the fixed points.

Denote $U = U_1 \cup U_2 \cup ... \cup U_t$, where U_i, $i = 1, ..., t$ is the neighborhood of the fixed point x_i.

Thus the trajectory firstly visits the neighborhood of a fixed point x_{n_1} then it visits the neighborhood of a fixed point x_{n_2} and so on.

The sequence $n_1, n_2, ...$ is called the itinerary (route-march) of the trajectory $x^{(0)}, x^{(1)}, ...$ Since the set of fixed points is a finite set, the numbers $n_1, n_2, ...$ will repeat.

Problem 3. *Is there a trajectory with periodic itinerary?*

On the basis of numerical calculations Ulam [44] conjectured that ergodic theorem holds for any QSO V, that is, the limit $\lim_{n\to\infty} \sum_{k=0}^{n} V^k(x)$ exists for any $x \in S^{m-1}$. In [45] Zakharevich proved that this conjecture is false in general. In [3] the authors established a necessary condition for the ergodic theorem to hold for the following class of Volterra QSOs $V : S^2 \to S^2$

$$\begin{aligned} x' &= x\,(1 + ay - bz)\,, \\ y' &= y\,(1 - ax + cz)\,, \\ z' &= z\,(1 + bx - cy)\,, \end{aligned} \tag{9}$$

where $a, b, c \in [-1, 1]$.

Note that if $a = b = c = 1$ the QSO (9) coincides with an example considered in [45].

Theorem 3. [3] *If the parameters a, b, c for the Volterra QSO (9) have the same sign and each is non-zero, then the ergodic theorem will fail for this operator.*

Problem 4. *Is the condition of Theorem 3 sufficient for the ergodic theorem to hold?*

Problem 5. *Find necessary and sufficient conditions on matrix A of Volterra QSO under which the ergodic theorem is true for the Volterra QSO on S^{m-1}, $m \geq 2$.*

4 The permuted Volterra QSO

Let τ be a cyclic permutation on the set of indices $1, 2, ..., m$ and let V be a Volterra QSO. Define QSO V_τ by

$$V_\tau : x'_{\tau(j)} = x_j \left(1 + \sum_{k=1}^{m} a_{jk} x_k\right), \; j = 1, ..., m, \tag{10}$$

where a_{jk} is defined in (8) (see [12],[14],[16],[19]).

Note that QSO V_τ is a non-Volterra QSO iff $\tau \neq$id.

Theorem 4. [14] *For any automorphism $W : S^{m-1} \to S^{m-1}$ there exists a permutation τ and a Volterra QSO V such that $W = V_\tau$.*

Corollary 1. *The set of all quadratic automorphisms of the simplex S^{m-1} can be geometrically presented as the union of $m!$ nonintersecting cubes of dimention $\frac{m(m-1)}{2}$.*

In [40] the behavior of trajectories of a non-Volterra automorphism $V : S^2 \to S^2$ are studied.

Problem 6. *Investigate the asymptotic behavior of the trajectories of the operators V_τ (automorphisms) for an arbitrary permutation τ.*

5 ℓ-Volterra QSO

Fix $\ell \in E$ and assume that elements $P_{ij,k}$ of the matrix $\mathbf{P}$ satisfy

$$P_{ij,k} = 0 \text{ if } k \notin \{i,j\} \text{ for any } k \in \{1,...,\ell\}, \quad i,j \in E; \tag{11}$$

$$P_{ij,k} > 0 \text{ for at least one pair } (i,j), \quad i \neq k, \quad j \neq k \text{ if } k \in \{\ell+1,...,m\}. \tag{11a}$$

Definition 2. For any fixed $\ell \in E$, the QSO defined by (4), (5), (11) and (11a) is called ℓ-Volterra QSO.

Denote by $\mathcal{V}_\ell$ the set of all ℓ-Volterra QSOs.

Remarks. 1. The condition (11a) guarantees that $\mathcal{V}_{\ell_1} \bigcap \mathcal{V}_{\ell_2} = \emptyset$ for any $\ell_1 \neq \ell_2$.
2. Note that ℓ-Volterra QSO is Volterra if and only if $\ell = m$.
3. By Theorem 2 we know that there is no a periodic trajectory for Volterra QSO. But for ℓ-Volterra QSO there is such trajectories (see Proposition 1 below).

Let $e_i = (\delta_{1i}, \delta_{2i}, ..., \delta_{mi}) \in S^{m-1}$, $i = 1,...,m$ be the vertices of S^{m-1}, where δ_{ij} the Kronecker delta.

Proposition 1.[41] 1) *For any set $I_s = \{e_{i_1}, ..., e_{i_s}\} \subset \{e_{\ell+1}, ..., e_m\}$, $s \leq m$, there exists a family $\mathcal{V}_\ell(I_s) \subset \mathcal{V}_\ell$ such that I_s is an s-cycle for every $V \in \mathcal{V}_\ell(I_s)$.*
2) For any $I_1, ..., I_q \subset \{\ell+1, ..., m\}$ such that $I_i \cap I_j = \emptyset$ $(i \neq j, i, j = 1, ..., q)$, there exists a family $\mathcal{V}_\ell(I_1, ..., I_q) \subset \mathcal{V}_\ell$ such that $\{e_i, i \in I_j\}$ $(j = 1, ..., q)$ is a $|I_j|$-cycle for every $V \in \mathcal{V}_\ell(I_1, ..., I_s)$.

Problem 7. *Find the set of all periodic trajectories of a given $\ell-$Volterra QSO.*

In paper [41] the trajectories of an 1-Volterra and 2-Volterra QSOs are studied.

Problem 8. *Develop theory of dynamical systems generated by a ℓ-Volterra QSO. Find its Lyapunov functions, the set of limit points of its trajectories etc.*

Note that in [4] a quasi-Volterra QSO was considered, such a QSO is a particular case of ℓ-Volterra QSO.

6 Non-Volterra QSO as combination of a Volterra and a non-Volterra operators

In [15] it was considered the following family of QSOs $V_\lambda : S^2 \to S^2$: $V_\lambda = \lambda V_0 + (1-\lambda)V_1, 0 \leq \lambda \leq 1$, where $V_0(x) = (x_1^2 + 2x_1x_2, x_2^2 + 2x_2x_3, x_3^2 + 2x_1x_3)$ is a Volterra QSO and $V_1(x) = (x_1^2 + 2x_2x_3, x_2^2 + 2x_1x_3, x_3^2 + 2x_1x_2)$ is a non-Volterra one.
Note that behavior of the trajectories of V_0 is very irregular (see [25], [45]). It has fixed points $M_0 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, e_1, e_2, e_3. The point M_0 is a repelling and $e_i, i = 1, 2, 3$ are saddle

points. These four points, are also fixed points for V_1 but M_0 is an attracting point for V_1. Thus properties of V_λ change depending on the parameter λ. In [15] some examples of invariant curves and the set of limit points of the trajectories of V_λ are given.

Problem 9. *For arbitrary two QSOs V_1 and V_2 connect the properties of $V_\lambda = \lambda V_1 + (1-\lambda)V_2$, $\lambda \in [0,1]$ with properties of V_1 and V_2.*

7 F-QSO

Consider $E_0 = E \cup \{0\} = \{0, 1, ..., m\}$. Fix a set $F \subset E$ and call this set the set of "females" and the set $M = E \setminus F$ is called the set of "males". The element 0 will play the role of an "empty-body".

Coefficients $P_{ij,k}$ of the matrix $\mathbf{P}$ we define as follows

$$P_{ij,k} = \begin{cases} 1, & \text{if} \quad k = 0, i, j \in F \cup \{0\} \quad \text{or} \quad i, j \in M \cup \{0\}; \\ 0, & \text{if} \quad k \neq 0, i, j \in F \cup \{0\} \quad \text{or} \quad i, j \in M \cup \{0\}; \\ \geq 0, & \text{if} \quad i \in F, j \in M, \forall k. \end{cases} \tag{12}$$

The biological interpretation of the coefficients (12) is obvious: the "child" k can be born only if its parents are taken from different classes F and M. Generally, $P_{ij,0}$ can be strictly positive for $i \in F$ and $j \in M$, which corresponds, for example, to the case in which "female" i with "male" j cannot have a "child", because one of them is ill or both are.

Definition 3. *For any fixed $F \subset E$, the QSO defined by (4),(5) and (12) is called the $F-$ quadratic stochastic operator (F-QSO).*

Remarks. 1. Any $F-$ QSO is non- Volterra, because $P_{ii,0} = 1$ for any $i \neq 0$.

2. For $m = 1$ there is a unique $F-$QSO (independently of $F = \{1\}$ and $F = \emptyset$) which is constant i.e., $V(x) = (1, 0)$ for any $x \in S^1$.

Theorem 5. [39] *Any F-QSO has a unique fixed point* $(1, 0, ..., 0)$ *(with m zeros). Besides, for any $x^0 \in S^m$, the trajectory $\{x^{(n)}\}$ tends to this fixed point exponentially rapidly.*

Problem 10. *Consider a partition $\xi = \{E_1, ..., E_q\}$ of E i.e., $E = E_1 \cup ... \cup E_q$, $E_i \cap E_j = \emptyset$, $i \neq j$. Assume $P_{ij,k} = 0$ if $i, j \in E_p$, for $p = 1, ..., q$. Call the corresponding operator a $\xi-$QSO. Is an analogue of Theorem 5 true for any $\xi-$QSO?*

8 Strictly non-Volterra QSO

Recently in [40] a new class of non-Volterra QSOs have been introduced. These QSOs satisfy

$$P_{ij,k} = 0 \;\; \text{if} \;\; k \in \{i, j\}, \;\; \forall i, j, k \in E. \tag{13}$$

Such an operator is called strictly non-Volterra QSO. One can easily check that the strictly non-Volterra operators exist only for $m \geq 3$.

An arbitrary strictly non-Volterra QSO defined on S^2 (i.e., $m = 3$) has the form:

$$\begin{aligned} x' &= \alpha y^2 + cz^2 + 2yz, \\ y' &= ax^2 + dz^2 + 2xz, \\ z' &= bx^2 + \beta y^2 + 2xy, \end{aligned} \tag{14}$$

where

$$a, b, c, d, \alpha, \beta \geq 0, \quad a + b = c + d = \alpha + \beta = 1. \tag{15}$$

Theorem 6. [40] 1) *For any values of parameters $a, b, c, d, \alpha, \beta$ with (15) the operator (14) has a unique fixed point. Moreover the fixed point is not attractive.*

2) *The QSO (14) has 2-cycles and 3-cycles depending on the parameters (15).*

Problem 11. *Is Theorem 6 true for $m \geq 4$?*

9 Regularity of QSO

In [17] the authors consider an arbitrary QSO $V : S^{m-1} \to S^{m-1}$ with matrix $\mathbf{P} = (P_{ij,k})$ and studied the problem of finding the smallest α_m such that the condition $P_{ij,k} > \alpha_m$ implies the regularity of V.

Theorem 7. [17] 1) *If $P_{ij,k} > \frac{1}{2m}$ then V is regular.*

2) $\alpha_2 = \frac{1}{2}(3 - \sqrt{7})$.

Problem 12. *Find exact values of α_m for any $m \geq 3$.*

10 Quadratic bistochastic operators

Let $x \in S^{m-1}$. Denote by $x_{\downarrow}$ the point $x_{\downarrow} = (x_{[1]}, ..., x_{[m]}) \in S^{m-1}$, where $x_{[1]} \geq ... \geq x_{[m]}$ are the coordinates of x in non-increasing order.

If $x, y \in S^{m-1}$ and

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \ \ k = 1, ..., m,$$

then we say that y majorizes x and write $x \prec y$.

As is known [27], $x \prec y$ iff there is a doubly stochastic (bistochastic) matrix B such that $x = By$. Therefore, if B is a bistochastic matrix, then $Bx \prec x$ for any point $x \in S^{m-1}$.

In [21] it was considered more general definition:

Definition 4. An arbitrary continuous operator $V : S^{m-1} \to S^{m-1}$ satisfying the condition

$$V(x) \prec x, \quad x \in S^{m-1} \tag{16}$$

is called a bistochastic operator.

Theorem 8. [21],[22] 1) *If $V : S^{m-1} \to S^{m-1}$ is a bistochastic operator, then the coefficients $P_{ij,k}$ satisfy the conditions*

$$\sum_{i,j=1}^{m} P_{ij,k} = m, \quad \forall k = 1, ..., m; \tag{a}$$

$$\sum_{j=1}^{m} P_{ij,k} \geq \frac{1}{2}, \quad \forall i, k = 1, ..., m; \tag{b}$$

$$\sum_{i,j \in I} P_{ij,k} \leq t, \quad \forall t, k = 1, ..., m, \tag{c}$$

where $I = \{i_1, ..., i_t\}$ is an arbitrary subset of $\{1, .., m\}$ containing t elements.

2) *If (c) holds for a QSO V then it is a bistochastic.*

Let $\mathbf{B}$ be the set of all bistochastic quadratic operators acting in S^{m-1}. The set $\mathbf{B}$ can be regarded as a polyhedron in an $\frac{m(m^2-1)}{2}$-dimensional space. Let Extr$\mathbf{B}$ be the set of extreme points of $\mathbf{B}$.

Theorem 9.[20],[21] *If $V \in$Extr$\boldsymbol{B}$, then*

$$P_{ii,k} = 0 \quad \text{or} \quad 1; \tag{d}$$

$$P_{ij,k} = 0, \frac{1}{2} \quad \text{or} \quad 1, \quad \text{for} \quad i \neq j. \tag{d}$$

Note that the converse assertion of the Theorem is false [21].

In [22] an analogue of Birkhoff's theorem is proved.

Problem 13. *Investigate the behavior of trajectories of the bistochastic quadratic operators.*

11 Surjective QSOs

In [5] and [28] a description of surjective QSOs defined on S^{m-1} for $m = 2, 3, 4$ and classification of extreme points of the set of such operators are given.

Problem 14. *Describe the set of all surjective QSOs defined on S^{m-1} for any $m \geq 5$.*

12 Construction of QSO

In papers [6],[7] a constructive description of $\mathbf{P}$ (i.e. QSO) is given. The construction depends on cardinality of E, namely two cases: (i) E is finite (ii) E is a continual set were separately considered. Note that for the second case one of the key problem is to determine the set of coefficients of heredity which is already infinite dimensional; the second problem is to investigate the quadratic operator which corresponds to this set of coefficients. By the construction the operator V depends on a probability measure μ being defined on a measurable space $(E, \mathcal{F})$.

Recall the construction for finite $E = \{1, ..., m\}$.

Let $G = (\text{Ł}, L)$ be a finite graph without loops and multiple edges, where Ł is the set of vertexes and L is the set of edges of the graph.

Furthermore, let Φ be a finite set, called the set of alleles (in problems of statistical mechanics, Φ is called the range of spin). The function $\sigma : \text{Ł} \to \Phi$ is called a cell (in mechanics it is called configuration). Denote by Ω the set of all cells, this set corresponds to E. Let $S(\text{Ł}, \Phi)$ be the set of all probability measures defined on the finite set Ω.

Let $\{\text{Ł}_i, i = 1, ..., q\}$ be the set of maximal connected subgraphs (components) of the graph G. For any $M \subset \text{Ł}$ and $\sigma \in \Omega$ denote $\sigma(M) = \{\sigma(x) : x \in M\}$. Fix two cells $\sigma_1, \sigma_2 \in \Omega$, and put

$$\Omega(G, \sigma_1, \sigma_2) = \{\sigma \in \Omega : \sigma(\text{Ł}_i) = \sigma_1(\text{Ł}_i) \quad \text{or} \quad \sigma(\text{Ł}_i) = \sigma_2(\text{Ł}_i) \quad \text{for} \quad \text{all} \quad i = 1, ..., m\}.$$

Now let $\mu \in S(\text{Ł}, \Phi)$ be a probability measure defined on Ω such that $\mu(\sigma) > 0$ for any cell $\sigma \in \Omega$; i.e μ is a Gibbs measure with some potential [37]. The heredity coefficients $P_{\sigma_1\sigma_2,\sigma}$

are defined as

$$P_{\sigma_1\sigma_2,\sigma} = \begin{cases} \frac{\mu(\sigma)}{\mu(\Omega(G,\sigma_1,\sigma_2))}, & \text{if} \ \ \sigma \in \Omega(G,\sigma_1,\sigma_2), \\ 0 \ \ \text{otherwise}. \end{cases} \tag{17}$$

Obviously, $P_{\sigma_1\sigma_2,\sigma} \geq 0$, $P_{\sigma_1\sigma_2,\sigma} = P_{\sigma_2\sigma_1,\sigma}$ and $\sum_{\sigma\in\Omega} P_{\sigma_1\sigma_2,\sigma} = 1$ for all $\sigma_1, \sigma_2 \in \Omega$.

The QSO $V \equiv V_\mu$ acting on the simplex $S(Ł, \Phi)$ and determined by coefficients (17) is defined as follows: for an arbitrary measure $l \in S(Ł, \Phi)$, the measure $V(l) = l' \in S(Ł, \Phi)$ is defined by the equality

$$l'(\sigma) = \sum_{\sigma_1,\sigma_2\in\Omega} P_{\sigma_1\sigma_2,\sigma} l(\sigma_1) l(\sigma_2) \tag{18}$$

for any cell $\sigma \in \Omega$.

Theorem 10. [6] *The QSO (18) is Volterra if and only if the graph G is connected.*

Thus if Φ, G and μ are given then we can constuct a QSO corresponding to these objects. In [6],[29] several examples of Φ, G and μ are considered and the trajectories of corresponding QSOs are studied.

Note that the construction above does not give all possible QSOs. So the following problem is interesting.

Problem 15. *Describe the class of QSOs which can be obtained by the construction.*

In [7] also constructively described QSOs which act to the set of all probability measures on some measurable space $(E, \mathcal{F})$ where E is a uncountable set. This construction depends on a Gibbs measure μ (see [37]). The behavior of trajectories of such operators were studied. These investigations allows to a natural introduction of thermodynamics in studying some models of heredity. More precisely, if E is continual set then one can associate the Gibbs measure μ by a Hamiltonian H (defined on E) and temperature $T > 0$ [37]. It is known that depending on the Hamiltonian and the values of the temperature the measure μ can be non unique. In this case there is a phase transition of the physical system with the Hamiltonian H.

In [7] for q-state Potts Hamiltonian when the temperature is low enough, it is proven that any trajectory of the QSO constructed by a Gibbs measure μ_i, $i = 1, ..., q$ of the Potts Hamiltonian tends to the measure μ_i. In other words, any trajectory of the QSO generated by a Gibbs measure of the Potts model converges to this measure.

Problem 16. (by N.N.Ganikhodjaev) *How the thermodynamics (the phase transition) will effect to behavior of the trajectories of a QSO corresponding to a Gibbs measure of the Hamiltonian H?*

13 Non-Volterra QSO generated by a product measure

In [38] it was shown that if μ is the product of probability measures being defined on each maximal connected subgraphs of G then corresponding non-Volterra operator can be reduced to q number (where q is the number of maximal connected subgraphs of G) of Volterra operators defined on the maximal connected subgraphs.

Let $G = (Ł, L)$ be a finite graph and $\{Ł_i, i = 1, ..., q\}$ the set of all maximal connected subgraphs of G. Denote by $\Omega_i = \Phi^{Ł_i}$ the set of all configurations defined on $Ł_i$, $i = 1, ..., q$. Let μ_i be a probability measure defined on Ω_i, such that $\mu_i(\sigma) > 0$ for any $\sigma \in \Omega_i$, $i = 1, ..., q$.

Consider probability measure μ on $\Omega = \Omega_1 \times \ldots \times \Omega_q$ defined by

$$\mu(\sigma) = \prod_{i=1}^{q} \mu_i(\sigma_i), \tag{19}$$

where $\sigma = (\sigma_1, ..., \sigma_q)$, with $\sigma_i \in \Omega_i, i = 1, ..., q$.

According to Theorem 10 if $q = 1$ then QSO constructed on G is Volterra QSO.

Theorem 11. [38] *The QSO constructed by the construction (18) with respect to measure (19) is reducible to q separate Volterra QSOs.*

This result allows to study a wide class of non-Volterra operators in the framework of the well known theory of Volterra quadratic stochastic operators.

Problem 17. *Describe the set of all non-Volterra QSOs which are reducible to several Volterra QSOs.*

Problem 18. *Find a measure μ different from (19) such that the non-Volterra QSO corresponding to μ can be investigated in the framework of a well known theory of QSOs.*

14 Trajectories with historic behavior

The problem which we shall discuss here is a particular case of the problem stated in [43].

Consider a QSO $V : S^{m-1} \to S^{m-1}$. We say that a trajectory $\{x, V(x), V^2(x), ...\}$ has historic behavior if for some continuous function $f : S^{m-1} \to R$ the average

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} f(V^i(x))$$

does not exist.

If this limit does not exist, it follows that "partial averages" $\frac{1}{n+1}\sum_{i=0}^{n} f(V^i(x))$ keep changing considerable so that their values give information about the epoch to which n belongs: they have a history [43].

Problem 19. *Find a class of QSOs such that the set of initial states which give rise to trajectories with historic behavior has positive Lebesque measure.*

Similar problem was discussed by Ruelle in [40].

15 A generalization of Volterra QSO

Consider QSO (4), (5) with additional condition

$$P_{ij,k} = a_{ik}b_{jk}, \quad \forall i, j, k \in E \tag{20}$$

where $a_{ik}, b_{jk} \in R$ entries of matrices $A = (a_{ik})$ and $B = (b_{jk})$ such that conditions (5) are satisfied for the coefficients (20).

Then the QSO V corresponding to the coefficients (20) has the form

$$x'_k = (V(x))_k = (A(x))_k \cdot (B(x))_k, \tag{21}$$

where

$$(A(x))_k = \sum_{i=1}^{m} a_{ik}x_i, \quad (B(x))_k = \sum_{j=1}^{m} b_{jk}x_j.$$

Note that if A (or B) is the identity matrix then operator (21) is a Volterra QSO.

Problem 20. *Develop theory of QSOs defined by (21).*

16 Bernstein's problem

The Bernstein problem [25],[26] is related to a fundamental statement of population genetics, the so-called stationarity principle. This principle holds provided that the Mendel law is assumed, but it is consistent with other mechanisms of heredity. An adequate mathematical problem is as follows. QSO V is a Bernstein mapping if $V^2 = V$. This property is just the stationarity principle. This property also is known as Hardy-Weinberg law [23]. The problem is to describe all Bernstein mappings explicitly. The case $m \leq 2$ is mathematically trivial and biologically not interesting. Bernstein [1] solved the above problem for the case $n = 3$ and obtained some results for $n \geq 4$. In works by Lyubich (see e.g.[25],[26]) the Bernstein problem was solved for all m under the regularity assumption. The regularity means that $V(x)$ depends only on the values $f(x)$, where f runs over all invariant linear forms. In investigations by Lyubich [25], the algebra A_V with the structure constants $P_{ij,k}$ played a very important role. Since $V(x) = x^2$, the Bernstein property of V is equivalent ti the identity

$$(x^2)^2 = s^2(x)x^2.$$

This identity means that A_V is a Bernstein algebra with respect to the algebra homomoprphism $s : A_V \to R$. The mapping V is regular iff the identity

$$x^2y = s(x)xy$$

holds in the algebra A_V, by definition, this identity means that A_V is regular.

Problem 21. *Describe all QSOs which satisfy $V^r(x) = V(x)$ for any $x \in S^{m-1}$ and some $r \geq 2$.*

17 Topological conjugacy

Definition 5. Let $V_1 : S^{m-1} \to S^{m-1}$ and $V_2 : S^{m-1} \to S^{m-1}$ be two QSOs with coefficients $P^{(1)}_{ij,k}$ and $P^{(2)}_{ij,k}$ respectively. V_1 and V_2 are said to be topologically conjugate if there exists a homeomorphism $h : S^{m-1} \to S^{m-1}$ such that, $h \circ V_1 = V_2 \circ h$. The homeomorphism h is called a topological conjugacy.

Mappings which are topologically conjugate are completely equivalent in terms of their dynamics [2].

Definition 6. A polynomial $f(P_{ij,k})$ is called an indicator if from the topologically conjugateness of V_1 and V_2 it follows that

$$\alpha_f \leq f(P^{(1)}_{ij,k}) \leq \beta_f \quad \text{and} \quad \alpha_f \leq f(P^{(2)}_{ij,k}) \leq \beta_f,$$

where $\alpha_f, \beta_f \in R$.

Definition 7. A system $f_1, ..., f_t$ of indicators is called complete if from

$$\alpha_{f_n} \leq f_n(P^{(1)}_{ij,k}) \leq \beta_{f_n} \quad \text{and} \quad \alpha_{f_n} \leq f_n(P^{(2)}_{ij,k}) \leq \beta_{f_n},$$

for any $n = 1, ..., t$ it follows the topologically conjugateness of V_1 and V_2.

A minimal complete system of indicators is called a basis.

Problem 22. *Does there exist a finite complete system of indicators? Find the basis of the system of indicators.*

Remark. There are many papers devoted to (quantum) quadratic stochastic processes [8],[9],[31]-[33] and to an infinite dimensional Volterra quadratic operators [34], [35]. Even exotic directions, such as the p-adic QSOs has been considered in [24]. Many other problems also discussed in [30].

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